EINSTEIN'S RECENT THEORY OF GRAVITATION AND ELECTRICITY

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1. In proposing his recent theory of gravitation and electricity Einstein[1] has derived his equations by expressing that a certain integral is stationary for the variations of a covariant tensor density of the second order and the coefficient of an asymmetrical connection. In this note we show more particularly what kind of a linear connection Einstein has employed and obtain in tensor form the equations which in this theory should replace Maxwell's equations.

2. Let a_{ij} be the components of a covariant tensor of the second order for which the determinant $a = |a_{ij}|$ is different from zero and denote by a^{ij} the cofactor of a_{ij} in a divided by a, then

$$a^{ij}a_{ij} = a^{ji}a_{ki} = \delta^j_k, \tag{2.1}$$

where δ^j_k is 1 or 0 according as $j = k$ or $j \neq k$. We put

$$A^{ij} = \sqrt{ea}\, a^{ij}, \tag{2.2}$$

where e is plus or minus 1 so that the radical is real; as thus defined A^{ji} is a tensor density.

Einstein's equations are (to within a change in notation)

$$L_{jk} \equiv \frac{\partial L^i_{ji}}{\partial x^k} - \frac{\partial L^i_{jk}}{\partial x^i} + L^h_{ji}\,L^i_{hk} - L^h_{jk}\,L^i_{hi} = 0, \tag{2.3}$$

and

$$\frac{\partial A^{ij}}{\partial x^k} + A^{ih}\,L^j_{kh} + A^{hj}\,L^i_{hk} - A^{ij}\,L^h_{kh} - \delta^j_k\left(\frac{\partial A^{ih}}{\partial x^h} + A^{hl}\,L^i_{hl}\right) = 0, \tag{2.4}$$

where L^i_{jk} are the coefficients of the linear connection.

Contracting (2.4) for i and k and j and k, we get

$$\frac{\partial A^{ij}}{\partial x^i} - \frac{\partial A^{ji}}{\partial x^i} = 0, \tag{2.5}$$

$$\frac{\partial A^{ih}}{\partial x^h} + A^{hl}\,L^i_{hl} = A^{ih}\lambda_h, \tag{2.6}$$

where

$$\lambda_h = \frac{1}{n-1}\,(L^j_{jh} - L^j_{hj}). \tag{2.7}$$

If in (2.4) we replace A^{ij} by their expressions (2.2) and make use of (2.6), we obtain

$$\frac{\partial a^{ij}}{\partial x^k} + a^{ij}\frac{\partial \log \sqrt{ea}}{\partial x^k} + a^{ih}L^j_{kh} + a^{hj}L^i_{hk} - a^{ij}L^h_{kh} - \delta^j_k a^{ih}\lambda_h = 0. \tag{2.8}$$

Multiplying by a_{ij}, summing for i and j and making use of the identity

$$\frac{\partial \log a}{\partial x^k} = -a_{ij}\frac{\partial a^{ij}}{\partial x^k},$$

we obtain (since $n \neq 2$)

$$\frac{\partial \log \sqrt{ea}}{\partial x^k} - L^h_{kh} + \lambda_h = 0. \tag{2.9}$$

Hence equations (2.8) become

$$\frac{\partial a^{ij}}{\partial x^k} + a^{ih}L^j_{kh} + a^{hj}L^i_{hk} - (a^{ij}\lambda_k + \delta^j_k \quad a^{ih}\lambda_h) = 0. \tag{2.10}$$

In consequence of (2.1) equations (2.10) can be written

$$\frac{\partial a_{ij}}{\partial x^k} - a_{ih}L^h_{kj} - a_{hj}L^h_{ik} + (a_{ij}\lambda_k + a_{ik}\lambda_j) = 0. \tag{2.11}$$

3. We put

$$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}, \tag{3.1}$$

where Γ^i_{jk} and Ω^i_{jk} are the symmetric and skew-symmetric parts, respectively, of L^i_{jk}. If we put

$$\Omega_j = \Omega^i_{ij} = -\Omega^i_{ji},$$

we have from (2.7)

$$\lambda_h = \frac{2}{n-1}\Omega_h.$$

If we put

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^h_{jl}\Gamma^i_{hk} - \Gamma^h_{jk}\Gamma^i_{hl},$$

and denote by b_{jk} and φ_{jk} the symmetric and skew-symmetric parts of the tensor $B_{jk} = B^i_{jki}$, we have

$$b_{jk} = \frac{1}{2}\left(\frac{\partial \Gamma^i_{ij}}{\partial x^k} + \frac{\partial \Gamma^i_{ik}}{\partial x^j}\right) - \frac{\partial \Gamma^i_{jk}}{\partial x^i} + \Gamma^i_{hj}\Gamma^h_{ik} - \Gamma^i_{ih}\Gamma^h_{jk},$$

$$\beta_{jk} = \frac{1}{2}\left(\frac{\partial \Gamma^i_{ij}}{\partial x^k} - \frac{\partial \Gamma^i_{ik}}{\partial x^j}\right). \tag{3.2}$$

When the expressions (3.1) are substituted in (2.3), we obtain

$$L_{jk} = b_{jk} + \beta_{jk} + \Omega_{jk}, \tag{3.3}$$

where

$$\Omega_{jk} = \Omega^i_{ji,k} - \Omega^i_{jk,i} + \Omega^h_{ji}\,\Omega^i_{hk} + \Omega^h_{jk}\,\Omega_h.$$

Here, and in what follows, a subscript preceded by a comma indicates covariant differentiation with respect to the Γ's; the expressions for this covariant differentiation are obtained from those of Riemannian geometry by replacing Christoffel symbols of the second kind by Γ's with the same indices.

4. In order to bring his equations into agreement with experimental results, Einstein makes the restriction

$$\lambda_i = \frac{2}{n-1}\,\Omega_i = 0. \tag{4.1}$$

If we put

$$a_{ij} = g_{ij} + \varphi_{ij}, \qquad a^{ij} = h^{ij} + \psi^{ij}, \tag{4.2}$$

where g_{ij} and h^{ij} are symmetric, and φ_{ij} and ψ^{ij} are skew-symmetric, equations (2.5) become

$$\frac{\partial\sqrt{ae}\,\psi^{ij}}{\partial x^i} = 0. \tag{4.3}$$

Moreover, equations (2.11) are equivalent to

$$g_{ij,k} = \varphi_{ih}\,\Omega^h_{kj} + \varphi_{jh}\,\Omega^h_{ki} \tag{4.4}$$

$$\varphi_{ij,k} = g_{ih}\,\Omega^h_{kj} - g_{hj}\,\Omega^h_{ki} \tag{4.5}$$

From equations (4.5) it follows that the tensor Ω^h_{ij} is given by

$$\Omega^h_{ij} = \tfrac{1}{2}g^{hk}(\varphi_{ij,k} - \varphi_{jk,i} - \varphi_{ki,j}), \tag{4.6}$$

where g^{hk} is defined by

$$g^{hk}\,g_{ik} = \delta^h_i.$$

Hence the conditions (4.1) impose the conditions

$$g^{ik}\,\varphi_{ij,k} = 0. \tag{4.7}$$

From (2.9), (3.1) and (4.1) we have

$$\frac{\partial \log\sqrt{ea}}{\partial x^k} = \Gamma^h_{hk}. \tag{4.8}$$

Consequently the tensor β_{jk} defined by (3.2) is zero, and if we equate to zero the symmetric and skew-symmetric parts of L_{jk}, equations (3.3) are equivalent to

$$b_{jk} + \Omega^h_{ji}\,\Omega^i_{hk} = 0 \tag{4.9}$$

and

$$\Omega^i_{jk,i} = 0. \tag{4.10}$$

5. If in (2.10) we substitute for a^{ij} from (4.2) and make use of (4.1), we get

$$h^{ij}{}_{,k} + \psi^{ir}\Omega^j_{kr} + \psi^{jr}\Omega^i_{kr} = 0 \tag{5.1}$$

and

$$\psi^{ij}{}_{,k} + h^{ir}\Omega^j_{kr} - h^{rj}\Omega^i_{kr} = 0. \tag{5.2}$$

Evidently these equations are equivalent to (4.4) and (4.5).

Contracting (5.2) for i and k, we have

$$\frac{\partial\psi^{ij}}{\partial x^i} + \psi^{rj}\,\Gamma^i_{ir} = 0.$$

Consequently equations (4.3) and (4.8) are equivalent.

Gathering together the preceding results, we see that the field is defined by a symmetric tensor g_{ij}, a skew-symmetric tensor φ_{ij} and a set of functions Γ^h_{ij} symmetric in i and j subject to the conditions (4.4), (4.7), (4.8), (4.9) and (4.10), where Ω^k_{ij} is defined in terms of these functions by (4.6). To these must be added necessarily the conditions of integrability of (4.4).

We can give another form to equations (4.10), if we introduce functions h_{ij} defined by

$$h_{ij}h^{ik} = \delta^k_j.$$

With the aid of these functions, it follows from (5.2) that

$$\Omega^k_{ij} = \tfrac{1}{2}(h_{ir}h_{js}h^{tk}\psi^{rs}{}_{,t} - h_{rj}\psi^{rk}{}_{,i} - \psi^{kr}{}_{,j}h_{ri})$$

and thus equations (4.10) are

$$(h_{ir}h_{js}h^{tk}\psi^{rs}{}_{,t} - h_{rj}\psi^{rk}{}_{,i} - \psi^{kr}{}_{,j}\,h_{ri})_{,k} = 0. \tag{5.3}$$

When we put

$$g_{ij} = -\delta_{ij} + \gamma_{ij},$$

where δ_{ij} is 1 or 0 as $i = j$ or $i \neq j$, and consider γ_{ij} and φ_{ij} as of the first order, the first order terms of equations (4.3) and (4.7) reduce to

$$\frac{\partial\varphi_{ij}}{\partial x^i} = 0$$

and of (5.3) to

$$\sum_k \frac{\partial}{\partial x^k}\left(\frac{\partial\varphi_{ij}}{\partial x^k} - \frac{\partial\varphi_{jk}}{\partial x^i} - \frac{\partial\varphi_{ki}}{\partial x^j}\right) = 0.$$

Thus (4.3), (4.7) and (5.3) are the equations in tensor form which give as first approximation the equations which Einstein proposes as a substitute for Maxwell's equations of empty space. To these must be added also equations (4.4) and (4.9).

When φ_{ij} are identified as the electric and magnetic intensities, they are not the components of the curl of a vector as in the classical theory, unless an additional condition is added.

[1] *Berlin Sitzungsberichte*, **22**, 1925, 414–420.

EVIDENCE FOR THE EXISTENCE OF ACTIVATED MOLECULES IN A CHEMICAL REACTION

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It is known that when ozone decomposes rapidly radiation is emitted, part of which at least lies in the visible and ultra-violet. Stuchtey[1] has photographed the spectrum of the radiation using a quartz spectrograph. The radiation is scattered across the visible and extends well out into the ultra-violet. In particular, the well known bands of ozone in the vicinity of 2450 Ångstrom units come in definitely in the spectrum of the luminescence of the decomposition; there is also radiation of still higher frequency extending to approximately 2200Å which seems to have been the limit of observation of Stuchtey's work.[2] It is the purpose of this article to give the probable explanation of the emission of this remarkably short wave-length radiation, which, as will be shown, requires for its excitation energy quantities much larger than can be afforded by the ordinary heat of reaction.

The heat of decomposition of ozone is well known, the best value being that given by Jahn,[3] who found 34,500 calories per mol. The decomposition can under no circumstances be in entirety a simple monomolecular process, for we cannot obtain an even number of oxygen atoms from one of ozone. The interaction of three molecules is impossible as the complete mechanism; for this also does not yield an even number of oxygen atoms. The interaction of four molecules, while leading to possible mechanisms, is so improbable that we shall not consider it here.

We hence conclude that the process leading to the decomposition of ozone involves the interaction of two molecules and thus would make available not more than the energy of decomposition of two molecules. This does not mean that the mechanism which determines the rate of